

# Two classical problems of contact mechanics

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In this introductory text, two key problems of contact mechanics are considered. The first of them (Section 1) is the problem of frictionless indentation of the parabolic stamp into a half-plane. Using this example, we show how to select the physically substantiated solution among the family of solutions depending on a parameter. The example of indentation of a flat stamp is derived as a limiting case of the problem for a rounded stamp.

In Section 2 we reproduce the treatment of the classic Hertz problem following the famous book by A.E.H.Love “*A treatise on the mathematical theory of elasticity*”, Fourth edition, Dover, New York, 1927.

## 1 A plane problem of frictionless contact

### 1.1 Formulation of the problem

Consider the problem of indentation of a rigid parabolic stamp into a half-plane (Fig. ??). We assume that the friction between the stamp and the half-plane is absent.

Let the profile of the stamp be

$$f(x) = \frac{x^2}{2R}, \quad (1)$$

where  $R$  is the radius of curvature of the parabola at the peak<sup>1</sup>. When a force of magnitude  $P$ , directed along the  $z$  axis, acts on the stamp, it is displaced in the positive  $z$  direction about the (unknown) value  $\delta$  and produces an unknown distribution of the contact pressure  $p(x) = -\sigma_z|_{z=0}$ . From symmetry considerations, the contact covers a segment  $(-a, a)$  of the  $x$ -axes. The quantity  $a$  is unknown beforehand.

To find the stress distribution, one must solve the two-dimensional Lamé equation with respect to the displacement vector  $\mathbf{u} = \mathbf{i}u_x(x, z) + \mathbf{k}u_z(x, z)$

$$(\lambda + 2\mu) \text{grad div } \mathbf{u} - \mu \text{rot rot } \mathbf{u} = 0, \quad -\infty < x < \infty, \quad z > 0 \quad (2)$$

(plane strain problem is considered). On the surface of the half-plane the following boundary conditions must be satisfied:

Under the stamp the elastic displacements should coincide with those of a rigid stamp (kinematic condition of contact):

$$u_z|_{z=0} = \delta - f(x) = \delta - \frac{x^2}{2R}, \quad -a < x < a, \quad (3)$$

and outside the stamp the half-plane boundary is free, i.e. the normal stresses vanish:

$$\sigma_z|_{z=0} = 0, \quad x < -a, \quad x > a. \quad (4)$$

Under the conditions of frictionless, or smooth contact, the tangential stresses vanish everywhere at the half-plane boundary:

$$\tau_{xz}|_{z=0} = 0, \quad -\infty < x < \infty. \quad (5)$$

The boundary-value problem (2)–(5) is called a *mixed* boundary-value problem, because the conditions (3) and (4) prescribe boundary values of unknown functions and their derivatives, respectively, on different parts of the boundary surface, in our case, the straight line  $z = 0$ . It can be called a “one time mixed problem,” because the second boundary condition (5) is the same for all points of the boundary.

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<sup>1</sup>Any smooth curve can be represented via a formula like (1) in the vicinity of its maximum with an accuracy to the terms of order  $x^3$ .

The stamp is in static equilibrium under the simultaneous action of the pressing force  $P$  and the normal contact stresses. This leads to the equilibrium condition

$$\int_{-a}^a \sigma_z|_{z=0} dx = -P. \quad (6)$$

## 1.2 Derivation and solution of a singular integral equation

The solution of the problem formulated can be found in different ways, for instance, with the aid of Fourier integral transform. We may use, however, the already well-known general formulae relating the derivatives of the surface displacements with the surface tractions (formulae (J 2.25)<sup>2</sup>,  $b = a$ ):

$$\begin{aligned} \frac{\partial u_x}{\partial x} \Big|_{z=0} &= -\frac{(1-2\nu)(1+\nu)}{E} p(x) - \frac{2(1-\nu^2)}{\pi E} \int_{-a}^a \frac{q(s) ds}{x-s}, \\ \frac{\partial u_z}{\partial x} \Big|_{z=0} &= -\frac{2(1-\nu^2)}{\pi E} \int_{-a}^a \frac{p(s) ds}{x-s} + \frac{(1-2\nu)(1+\nu)}{E} q(x). \end{aligned} \quad (7)$$

Note that the quantity  $p$  in these formulae is the contact pressure, equal in magnitude to the normal stresses, taken with opposite sign,  $p(x) = -\sigma_z|_{z=0}$  and  $q(x) = \tau_{xz}|_{z=0}$  are the tangential stresses;  $\nu$  and  $E$  are Poisson's ratio and Young's modulus of the elastic material. These formulae are valid for all values of  $x$  from the interval  $-\infty < x < \infty$ .

In the case of smooth contact we have  $q(x) \equiv 0$  and we don't need to consider the first of equations (7). After the determination of the normal stresses, the tangential displacement is found simply. The second of these relations becomes

$$\frac{\partial u_z}{\partial x} \Big|_{z=0} = -\frac{2(1-\nu^2)}{\pi E} \int_{-a}^a \frac{p(s) ds}{x-s}, \quad -\infty < x < \infty. \quad (8)$$

If the contact pressure was known, we could determine the vertical displacement from the last equation. In our problem, on the contrary, the pressure is unknown and the left-hand side of (8) is known for  $-a < x < a$  on the base of (3). Substituting (3) into (8), we obtain a singular integral equation of the first kind with respect to the contact pressure (the integral is known as the Cauchy principal value integral),

$$\int_{-a}^a \frac{p(s) ds}{x-s} = \frac{\pi E}{2(1-\nu^2)} \frac{x}{R}, \quad -a < x < a. \quad (9)$$

Denoting the right-hand side of the last equation by  $g(x)$ , we use the general solution of this equation, given by a principal value integral as well (J 2.41)<sup>3</sup>:

$$p(x) = -\frac{1}{\pi^2 \sqrt{a^2 - x^2}} \int_{-a}^a \frac{\sqrt{a^2 - s^2}}{x-s} g(s) ds + \frac{C}{\pi^2 \sqrt{a^2 - x^2}}. \quad (10)$$

Here  $C$  is an arbitrary constant.

Substituting the actual expression for  $g(x)$  from (9), we write the general solution in the form

$$p(x) = \frac{1}{\sqrt{a^2 - x^2}} \left[ -\frac{E}{2\pi R(1-\nu^2)} \int_{-a}^a \frac{s\sqrt{a^2 - s^2}}{x-s} ds + C \right]. \quad (11)$$

Calculating the principal value integral using (J A1.3),

$$\int_{-a}^a \frac{s\sqrt{a^2 - s^2}}{x-s} ds = \frac{\pi}{2} (2x^2 - a^2), \quad -a < x < a, \quad (12)$$

we find

$$p(x) = \frac{1}{\sqrt{a^2 - x^2}} \left[ \frac{E}{4R(1-\nu^2)} (a^2 - 2x^2) + C \right]. \quad (13)$$

<sup>2</sup>Formulae marked with the letter "J" correspond to those from the book by K.L.Johnson "Contact Mechanics," Cambridge, 1985.

<sup>3</sup>There is a misprint in Johnson's book, the minus sign before the integral is missed.

Now, rewriting the static condition (6) in the form

$$\int_{-a}^a p(x) dx = P, \quad (14)$$

we find  $C = P/\pi$  and obtain

$$p(x) = \frac{1}{\sqrt{a^2 - x^2}} \left[ \frac{E}{4R(1 - \nu^2)} (a^2 - 2x^2) + \frac{P}{\pi} \right]. \quad (15)$$

### 1.3 Kinematics of deformation and analysis of solution

It is seen that though we have already fulfilled both the boundary conditions of the problem and the static condition, the size of the contact area  $a$  remains undetermined. It is characteristic for mixed boundary value problems for partial differential equations that the solution found is nonunique. However, there is an opportunity to specify a relevant value of  $a$  and to obtain a unique solution.

Let us introduce the quantity

$$a_0^2 = \frac{4PR(1 - \nu^2)}{\pi E} \quad (16)$$

and rewrite (15) in the form

$$p(x) = \frac{P}{\pi a_0^2} \left( 2\sqrt{a^2 - x^2} + \frac{a_0^2 - a^2}{\sqrt{a^2 - x^2}} \right). \quad (17)$$

We see that when  $a = a_0$ , the pressure distribution is represented by a bounded function

$$p(x) = \frac{2P}{\pi a_0^2} \sqrt{a_0^2 - x^2}, \quad (18)$$

hence it vanishes at the ends of the contact area. It is indeed the solution looked for and  $a = a_0$  is the half-length of the contact area. Any other values of  $a$  give unbounded distributions of the contact pressure. To understand the special role played by the solution (18) with respect to other (unbounded) solutions, let us consider the profile of the deformed surface, i.e., the function  $u_z|_{z=0}$ .

The relation (8) gives the slope of the half-plane boundary both inside the contact area and outside of it. In the former case it is represented by a principal value integral, while in the latter case it is a convenient integral; both of the cases can be evaluated by the formula

$$\frac{\partial u_z}{\partial x} \Big|_{z=0} = -\frac{2(1 - \nu^2)}{\pi E} \frac{P}{\pi a_0^2} \int_{-a}^a \left( 2\sqrt{a^2 - s^2} + \frac{a_0^2 - a^2}{\sqrt{a^2 - s^2}} \right) \frac{ds}{x - s}, \quad -\infty < x < \infty. \quad (19)$$

The integrals to be calculated are

$$\int_{-a}^a \frac{\sqrt{a^2 - s^2}}{x - s} ds = \pi \times \begin{cases} x + \sqrt{x^2 - a^2}, & x < -a, \\ x, & -a < x < a, \\ x - \sqrt{x^2 - a^2}, & x > a \end{cases} \quad (20)$$

and

$$\int_{-a}^a \frac{1}{\sqrt{a^2 - s^2}(x - s)} ds = \pi \times \begin{cases} -\frac{1}{\sqrt{x^2 - a^2}}, & x < -a, \\ 0, & -a < x < a, \\ \frac{1}{\sqrt{x^2 - a^2}}, & x > a. \end{cases} \quad (21)$$

The latter integral can be obtained by differentiation of the former with respect to the parameter  $a$ .

Substituting (20) and (21) into (19), we find

$$\frac{\partial u_z}{\partial x} \Big|_{z=0} = \begin{cases} -\frac{x}{R}, & 0 < x < a, \\ -\frac{1}{R} \left[ x - \sqrt{x^2 - a^2} + \frac{1}{2} \frac{a_0^2 - a^2}{\sqrt{x^2 - a^2}} \right], & x > a. \end{cases} \quad (22)$$

We see that (22) is a continuous function of  $x$  if and only if  $a = a_0$ . Otherwise, the slope of the free surface  $z = 0$ ,  $|x| > a$  undergoes a discontinuity of the sign corresponding to that of the difference  $a - a_0$ .

Integrating (22) over  $x$  and demanding that  $u_z|_{z=0}$  be  $\delta$  at  $x = 0$ , we obtain

$$u_z|_{z=0} = \begin{cases} \delta - \frac{x^2}{2R}, & 0 < x < a, \\ \delta - \frac{x^2}{2R} + \frac{1}{2R} \left( x\sqrt{x^2 - a^2} - a^2 \ln \frac{x + \sqrt{x^2 - a^2}}{a} \right) - \frac{a_0^2 - a^2}{2R} \ln \frac{x + \sqrt{x^2 - a^2}}{a}, & x > a. \end{cases} \quad (23)$$

The functions defined by (22) and (23) should be expanded for negative  $x$  as odd and even functions respectively.

The displacements for  $a = a_0$  are

$$u_z|_{z=0} = \begin{cases} \delta - \frac{x^2}{2R}, & 0 < x < a_0, \\ \delta - \frac{x^2}{2R} + \frac{1}{2R} \left( x\sqrt{x^2 - a_0^2} - a_0^2 \ln \frac{x + \sqrt{x^2 - a_0^2}}{a_0} \right) & x > a_0. \end{cases} \quad (24)$$

The quantity  $\delta$ , as in other plane problems of elastostatics, remains undetermined. We cannot determine it, say, from the requirement  $\lim_{x \rightarrow \infty} u_z|_{z=0} = 0$ , because it is seen from (23) and (24) that this limit is always  $-\infty$  for any finite value of  $\delta$ .

In the Fig. ??, (a) we depicted the results of analytical solution for the case  $a = a_0$ , given by formulae (17) for the contact pressure and (24) for the displacements. The results for  $a < a_0$  and  $a > a_0$  are presented in Fig. ??, (b) and Fig. ??, (c), respectively.

It is seen that the results in Fig. ??, (a) correspond to all physical requirements to the correct solution, even not formulated explicitly in Section 1.1: (i) The pressure  $p(x)$  is positive everywhere in the contact region and (ii) There is no intersection between the stamp and the half-plane outside the contact, i.e.,  $u_z|_{z=0} > f(x)$  for  $|x| > a$  (note the direction of  $z$ -axes).

The solution, presented in Fig. ??, (b), violates the condition (ii): We observe the intersection. The situation looks like we have the stamp cut along the dotted lines and therefore the contact area remains unchanged under subsequent increasing of the contact force  $P$ . This solution, hence, does not agree with the physical requirement that for a smooth stamp, the dependence of the contact area length should be a smooth function of the pressing force.

For the case (c), the contact pressure is negative in the vicinity of points  $x = \pm a$ . The shape of the free surface shows, that the solution with too big contact area violates the requirement of *unilateral* contact. The latter means, that the surfaces in contact should be *pressed* against each other.

## 1.4 Indentation of a flat stamp

An interesting and practically important case of the flat stamp can be obtained from the already found solution by the limiting transition  $R \rightarrow \infty$ . The profile of the stamp is represented by  $f(x) = \delta$  and passing to the limit in (17) and (23) we found the pressure

$$p(x) = \frac{P}{\pi\sqrt{a^2 - x^2}} \quad (25)$$

and the displacements of the half-plane boundary

$$u_z|_{z=0} = \begin{cases} \delta, & 0 < x < a, \\ \delta - \frac{2P(1 - \nu^2)}{E} \ln \frac{x + \sqrt{x^2 - a^2}}{a}, & x > a. \end{cases} \quad (26)$$

This solution is depicted in Fig. ??. Qualitatively, the pressure and displacement are similar to those from Fig. ??, (b). However, now the size of the contact area  $a$  is determined by the stamp geometry and unlike the solution for the rounded stamp, can not be adjusted according to any considerations. The discontinuity of the contact pressure is the natural consequence of the assumption that the stamp has sharp edges.

## 2 The Hertz problem

The material presented below corresponds to Sections 137–140 (pp. 193–200) of the book by Love. For convenience, the numeration of sections and equation correspond to those of original. In addition to Love’s footnotes, reproduced in Roman type, my additional footnotes are in Sans Serif type.

### 137. Pressure between two bodies in contact.—Geometrical Preliminaries

Let two bodies be pressed together so that the resultant pressure between them is  $P$ . The parts of the bodies near the points of contact will be compressed, so that there is contact over a small area of the surface of each. This common area will be called the *compressed area*, and the curve that bounds it the *curve of compression*. We propose to determine the curve of compression and the distribution of pressure over the compressed area<sup>4</sup>.

The shapes, in the unstressed state, of the two bodies near the parts that come into contact can be determined, with sufficient approximation, by equations of the form

$$\begin{aligned} z_1 &= A_1x^2 + B_1y^2 + 2H_1xy, \\ z_2 &= A_2x^2 + B_2y^2 + 2H_2xy, \end{aligned} \tag{40}$$

the axes of  $z_1$  and  $z_2$  being directed along the normals drawn towards the interiors of the bodies respectively. In the unstressed state, the bodies are in contact at the origin of  $(x, y)$ , they have a common tangent plane there, and the distance apart of two points of them, estimated along the common normal, is expressed with sufficient approximation by the quadratic form  $(A_1 + A_2)x^2 + (B_1 + B_2)y^2 + 2(H_1 + H_2)xy$ . This expression must be positive in whatever way the axes  $x$  and  $y$  are chosen, and we may choose these axes so that  $H_1 + H_2$  vanishes. Then  $A_1 + A_2$  and  $B_1 + B_2$  must be positive. We may therefore write

$$A_1 + A_2 = A, \quad B_1 + B_2 = B, \quad H_1 = -H_2, \tag{41}$$

$A$  and  $B$  being positive.

If  $R_1, R'_1$  are the principal radii of curvature at the point of contact for the body (1), and  $R_2, R'_2$  those for the body (2), and if these have positive signs when the corresponding centers of curvature are inside the bodies respectively, we have

$$2(A + B) = \frac{1}{R_1} + \frac{1}{R'_1} + \frac{1}{R_2} + \frac{1}{R'_2}. \tag{42}$$

The angle ( $\omega$ ) between those normal sections in which the radii of curvature are  $R_1, R_2$  is given by the equation

$$4(A - B)^2 = \left(\frac{1}{R_1} - \frac{1}{R'_1}\right)^2 + \left(\frac{1}{R_2} - \frac{1}{R'_2}\right)^2 + 2\left(\frac{1}{R_1} - \frac{1}{R'_1}\right)\left(\frac{1}{R_2} - \frac{1}{R'_2}\right)\cos 2\omega. \tag{43}$$

The angle ( $\omega'$ ) between the  $(x, z)$  plane, chosen so that  $H_2 = -H_1$ , and the normal section in which the radius of curvature is  $R_1$  is given by the equation

$$\left(\frac{1}{R_2} - \frac{1}{R'_2}\right)\sin 2(\omega - \omega') = \left(\frac{1}{R_1} - \frac{1}{R'_1}\right)\sin 2\omega'. \tag{44}$$

If we introduce an angle  $\tau$  by the equation

$$\cos \tau = \frac{B - A}{B + A}, \tag{45}$$

so that

$$2A \operatorname{cosec}^2 \frac{\tau}{2} = 2B \sec^2 \frac{\tau}{2} = \frac{1}{R_1} + \frac{1}{R'_1} + \frac{1}{R_2} + \frac{1}{R'_2}, \tag{46}$$

the shape of the “relative indicatrix,”  $Ax^2 + By^2 = \text{const.}$ , depends on the angle  $\tau$  only.

<sup>4</sup>The theory is due to Hertz, *J. f. Math. (Crelle)*, Bd. 92 (1881), reprinted in *Ges. Werke von Heinrich Hertz*, Bd. 1, Leipzig 1895, p. 155.

When the bodies are pressed together there will be displacement of both. We take the displacement of the body (1) to be  $(u_1, v_1, w_1)$  relative to the axes of  $(x, y, z_1)$ , and that of body (2) to be  $(u_2, v_2, w_2)$  relative to the axes  $(x, y, z_2)$ . Since the parts within the compressed area are in contact after the compression, we must have, at all points of this area,

$$z_1 + w_1 = -(z_2 + w_2) + \alpha,$$

where  $\alpha$  is the value of  $w_1 + w_2$  at the origin<sup>5</sup>. Hence within the compressed area we have

$$w_1 + w_2 = \alpha - Ax^2 - By^2, \quad (47)$$

and outside the compressed area we must have

$$w_1 + w_2 > \alpha - Ax^2 - By^2, \quad (48)$$

in order that the surfaces may be separated from each other.

### 138. Solution of the problem of the pressure between two bodies in contact

We denote by  $\lambda_1, \mu_1$  the elastic constants of the body (1), and by  $\lambda_2, \mu_2$  those of the body (2). The pressure  $P$  between the bodies is the resultant of a distributed pressure ( $p$  per unit of area)<sup>6</sup> over the compressed area. We may form functions  $\phi_1, \chi_1, \Omega_1$  for the body (1) in the same way as  $\phi, \chi, \Omega$  were formed in Article 136, and we may form corresponding functions for the body (2). The values of  $w_1$  and  $w_2$  at the common surface can then be written,

$$w_1 = \theta_1 \phi_0, \quad w_2 = \theta_2 \phi_0, \quad (49)$$

where<sup>7</sup>

$$\theta_1 = \frac{\lambda_1 + 2\mu_1}{4\pi\mu_1(\lambda_1 + \mu_1)} = \frac{1 - \nu_1^2}{\pi E_1}, \quad \theta_2 = \frac{\lambda_2 + 2\mu_2}{4\pi\mu_2(\lambda_2 + \mu_2)} = \frac{1 - \nu_2^2}{\pi E_2}, \quad (50)$$

and  $\phi_0$  is the value of  $\phi_1$  or  $\phi_2$  at the interface, i.e., the value of the convergent integral  $\iint pr^{-1} dx' dy'$  at a point of the surface. The value of  $\phi_0$  at any point within the compressed area is determined in terms of the quantity  $\alpha$  and the coordinates of the point by the equation

$$\phi_0 = \frac{1}{\theta_1 + \theta_2} (\alpha - Ax^2 - By^2). \quad (51)$$

This result suggests the next step in the solution of the problem<sup>8</sup>. The functions denoted by  $\phi_1$  and  $\phi_2$  are the potentials, on the two sides of the plane  $z = 0$ , of a superficial distribution of density  $p$  within

<sup>5</sup>If the points  $(x_1, y_1, z_1)$  of the body (1) and  $(x_2, y_2, z_2)$  of the body (2) come into contact, we must have

$$x_1 + u_1 = x_2 + u_2, \quad y_1 + v_1 = y_2 + v_2, \quad z_1 + w_1 = -(z_2 + w_2) + \alpha;$$

and in equation (47) we identify  $(x_1, y_1)$  with  $(x_2, y_2)$ . We may show that, without making this identification, we should have

$$w_1 + w_2 = \alpha - Ax_1^2 - By_1^2 - 2[A_2 \frac{1}{2}(x_1 - x_2)(u_1 - u_2) + B_2 \frac{1}{2}(y_1 - y_2)(v_1 - v_2) + H_2 \{x_1(v_1 - v_2) + y_1(u_1 - u_2)\}].$$

In the result we shall find for  $w_1 + w_2$  an expression of the order  $Aa^2$ , where  $a$  is the greatest diameter of the compressed area, and  $u_1, u_2, \dots$  will be of the same order as  $w_1 + w_2$ ; thus the terms neglected are of a higher order of small quantities than those retained. If the bodies are of the same material we have  $u_1 = u_2$  and  $v_1 = v_2$  when  $x_1 = x_2$  and  $y_1 = y_2$ , and thus the identification of  $(x_1, y_1)$  with  $(x_2, y_2)$  leads in this case to an exact result.

<sup>6</sup>Here Love's notation  $P'$  is replaced by  $p$ .

<sup>7</sup> $\nu_1, \nu_2, E_1, E_2$  are the Poisson's ratios and Young's moduli of the bodies.

<sup>8</sup>One may see that (49) can be expanded as

$$w_1(x, y) = \theta_1 \iint \frac{p(x', y') dx' dy'}{\sqrt{(x - x')^2 + (y - y')^2}}, \quad w_2(x, y) = \theta_2 \iint \frac{p(x', y') dx' dy'}{\sqrt{(x - x')^2 + (y - y')^2}}$$

and the combination of the last formulae with (47) gives the *integral equation*

$$\iint \frac{p(x', y') dx' dy'}{\sqrt{(x - x')^2 + (y - y')^2}} = \frac{1}{\theta_1 + \theta_2} (\alpha - Ax^2 - By^2),$$

which is the expanded (51). In this equation both the domain of integration and the function  $p$  are unknown. This is typical for about any contact problem.

the compressed area, and the potential at a point of this area is a quadratic function of the coordinates of the point. We recall the result that the potential of a homogeneous ellipsoid at an internal point is a quadratic function of the coordinates of the point<sup>9</sup>. We therefore seek to satisfy the conditions of the problem by assuming that the compressed area is the area within an ellipse, regarded as an ellipsoid very much flattened, and the pressure  $p$  may be obtained by a limiting process, the whole mass of the ellipsoid remaining finite, and one of its principal axes being diminished indefinitely. In the case of an ellipsoid of density  $\rho$ , whose equation referred to its principal axes is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the mass would be  $\frac{4}{3}\pi\rho abc$ ; the part of this mass that would be contained in a cylinder standing on the element of area  $dx'dy'$  would be

$$2\rho dx'dy'c\sqrt{1 - \frac{x'^2}{a^2} - \frac{y'^2}{b^2}},$$

and the potential at any external point would be

$$\pi\rho abc \int_{\nu}^{\infty} \left(1 - \frac{x^2}{a^2 + \psi} - \frac{y^2}{b^2 + \psi} - \frac{z^2}{c^2 + \psi}\right) \frac{d\psi}{\sqrt{(a^2 + \psi)(b^2 + \psi)(c^2 + \psi)}},$$

where  $\nu$  is the positive root of the equation

$$\frac{x^2}{a^2 + \nu} + \frac{y^2}{b^2 + \nu} + \frac{z^2}{c^2 + \nu} = 1.$$

At an internal point we should have the same form for the potential with 0 written for  $\nu$ . We have now to pass to a limit by taking  $c$  to diminish indefinitely, and  $\rho$  to increase indefinitely, while  $a$  and  $b$  remain finite, in such a way that

- (i)  $\frac{4}{3}\pi(\rho c)ab = P,$
- (ii)  $2(\rho c)\sqrt{1 - \frac{x'^2}{a^2} - \frac{y'^2}{b^2}} = p(x', y'),$
- (iii)  $\phi_0 = \pi ab(\rho c) \int_0^{\infty} \left(1 - \frac{x^2}{a^2 + \psi} - \frac{y^2}{b^2 + \psi}\right) \frac{d\psi}{\sqrt{(a^2 + \psi)(b^2 + \psi)\psi}},$

the third of these conditions being satisfied at all points within the compressed area. Hence we have

$$p(x', y') = \frac{3P}{2\pi ab} \sqrt{1 - \frac{x'^2}{a^2} - \frac{y'^2}{b^2}}, \quad (52)$$

and

$$\begin{aligned} & \frac{1}{\theta_1 + \theta_2}(\alpha - Ax^2 - By^2) \\ &= \frac{3}{4}P \int_0^{\infty} \left(1 - \frac{x^2}{a^2 + \psi} - \frac{y^2}{b^2 + \psi}\right) \frac{d\psi}{\sqrt{(a^2 + \psi)(b^2 + \psi)\psi}}. \end{aligned} \quad (53)$$

The equation (52) determines the law of distribution of the pressure  $p$  over the compressed area, when the dimensions of this area are known. The equation (53) must hold for all values of  $x$  and  $y$  within this area, and it is therefore equivalent to three equations, viz.

$$\begin{aligned} \alpha &= \frac{3}{4}P(\theta_1 + \theta_2) \int_0^{\infty} \frac{d\psi}{\sqrt{(a^2 + \psi)(b^2 + \psi)\psi}}, \\ A &= \frac{3}{4}P(\theta_1 + \theta_2) \int_0^{\infty} \frac{d\psi}{(a^2 + \psi)\sqrt{(a^2 + \psi)(b^2 + \psi)\psi}}, \\ B &= \frac{3}{4}P(\theta_1 + \theta_2) \int_0^{\infty} \frac{d\psi}{(b^2 + \psi)\sqrt{(a^2 + \psi)(b^2 + \psi)\psi}}. \end{aligned} \quad (54)$$

<sup>9</sup>Of course, this non-evident statement should be proved; the proof can be found in books about the theory of the potential, also called Newtonian potential.

The second and third of these equations determine  $a$  and  $b$ , and the first of them determines  $\alpha$  when  $a$  and  $b$  are known. If we express the results in terms of the eccentricity ( $e$ ) of the ellipse,  $e$  will be determined by the equation<sup>10</sup>

$$B \int_0^\infty \frac{d\zeta}{(1+\zeta)\sqrt{(1+\zeta)(1+\zeta-e^2)\zeta}} = A \int_0^\infty \frac{d\zeta}{(1+\zeta-e^2)\sqrt{(1+\zeta)(1+\zeta-e^2)\zeta}}, \quad (55)$$

$a$  will be given by the equation

$$Aa^3 = \frac{3}{4}P(\theta_1 + \theta_2) \int_0^\infty \frac{d\zeta}{(1+\zeta)\sqrt{(1+\zeta)(1+\zeta-e^2)\zeta}}, \quad (56)$$

and  $\alpha$  will be given by the equation

$$\alpha = \frac{3P}{4a}(\theta_1 + \theta_2) \int_0^\infty \frac{d\zeta}{\sqrt{(1+\zeta)(1+\zeta-e^2)\zeta}}. \quad (57)$$

We observe that  $e$  depends on the ratio  $A : B$  only. Hertz has tabulated the values of  $b/a = (1 - e^2)^{\frac{1}{2}}$ , in terms of the angle  $\tau$ , of which the cosine is  $(B - A)/(B + A)$ . He found the following results:

$\tau =$	90°	80°	70°	60°	50°	40°	30°	20°	10°	0°
$b/a =$	1	0.79	0.62	0.47	0.36	0.26	0.18	0.10	0.05	0

At points on the plane  $z = 0$  which are outside the compressed area,  $\phi_0$  is the potential, at external points in this plane, due to the distribution  $p$  over the compressed area. It follows from (49) that at points on the surfaces of the bodies, outside the compressed area and not far from it, we may write, with sufficient approximation

$$w_1 + w_2 = (\theta_1 + \theta_2) \frac{3P}{4} \int_\nu^\infty \left(1 - \frac{x^2}{a^2 + \psi} - \frac{y^2}{b^2 + \psi}\right) \frac{d\psi}{\sqrt{(a^2 + \psi)(b^2 + \psi)\psi}},$$

where  $\nu$  is the positive root of the equation

$$\frac{x^2}{a^2 + \nu} + \frac{y^2}{b^2 + \nu} = 1. \quad (58)$$

Hence we have

$$\begin{aligned} & (w_1 + w_2) - (\alpha - Ax^2 - By^2) \\ &= -(\theta_1 + \theta_2) \frac{3P}{4} \int_0^\nu \left(1 - \frac{x^2}{a^2 + \psi} - \frac{y^2}{b^2 + \psi}\right) \frac{d\psi}{\sqrt{(a^2 + \psi)(b^2 + \psi)\psi}}. \end{aligned} \quad (59)$$

Now, when  $\psi$  lies between 0 and  $\nu$ , the point  $(x, y)$ , which is on the ellipse (58), is outside the ellipse  $x^2/(a^2 + \psi) + y^2/(b^2 + \psi) = 1$ , and therefore the expression on the right-hand side of equation (59) is positive. The condition of inequality (48) is therefore satisfied.

The assumptions that the compressed area is bounded by an ellipse  $x^2/a^2 + y^2/b^2 = 1$ , where  $a$  and  $b$  are determined by the second and third of equations (54), and that the pressure  $p$  over this area is expressed by the formula (52), satisfy all the conditions of the problem. When  $p$  is known the functions  $\phi$ ,  $\chi$ ,  $\Omega$  for each of the bodies can be calculated, and hence we may determine the displacement and distribution of stress in each body.

Hertz<sup>11</sup> has drawn the lines of principal stresses in the  $(x, z)$  plane for the case in which  $\lambda = 2\mu$  (Poisson's ratio =  $\frac{1}{3}$ ). His drawing was in part conjectural, as the differential equation determining the directions of the lines of principal stress cannot be integrated exactly. A more exact result has been obtained by S.Fuchs<sup>12</sup>, by a method of approximate integration, in the case of a sphere resting on a plane. The lines of principal stresses in the body with the spherical boundary are represented in Fig. 15, where the full curved lines are lines of principal stress along which the traction is pressure, and the

<sup>10</sup>The eccentricity is determined as  $e = \sqrt{1 - b^2/a^2}$ , so that  $b^2 = a^2(1 - e^2)$ .

We perform the substitution of the integration variable  $\psi = a^2\zeta$ . The right-hand sides of the three equations below can be expressed through the complete elliptic integrals of the first and the second kind.

<sup>11</sup>*Verhandlungen des Vereins zur Beförderung des Gewerbefleisses*, 1882, reprinted in Hertz, *Ges. Werke*, Bd. 1, p. 174.

<sup>12</sup>*Physikalische Zeitschr.*, 1913, p. 1282. Further discussion of the case considered by Fuchs will be found in a paper by W.B.Morton and L.J.Close, *Phil. Mag.* (Ser. 6), vol. 43, 1922, p. 320.

dotted lines are lines of principal stress along which the traction is tension<sup>13</sup>. It will be observed that near the compressed area both the principal stresses are pressures. A little further away one set of lines shows tension near the surface and pressure in the central portions. Still further away the same set of lines shows tension throughout. The other set of lines are always lines of pressure.

Hertz made a series of experiments with the view of testing his theory. The result that the linear dimensions of the compressed area are proportional to the cube root of the pressure between the bodies was verified very exactly; the dependence of the form of the compressed area upon the form of the relative indicatrix was also verified in cases in which the latter could be determined with fair accuracy.

### 139. Hertz's theory of impact

The results obtained in the last Article have been applied to the problem of the impact of two solid bodies<sup>14</sup>. The ordinary theory of impact, founded by Newton, divides bodies into two classes, "perfectly elastic" and "imperfectly elastic." In the case of the former class there is no loss of kinetic energy in impact. In the other case energy is dissipated in impact. Many actual bodies are not very far from being perfectly elastic in the Newtonian sense. Hertz's theory of impact takes no account of the dissipation of energy; the compression at the place of contact is regarded as gradually produced and as subsiding completely by reversal of the process by which it is produced. The local compression is thus regarded as a statical effect. In order that such a theory may hold it is necessary that the duration of the impact should be a large multiple of the gravest period of free vibration of either body which involves compression at the place in question. A formula for the duration of the impact, which satisfies this requirement when the bodies impinge on each other with moderate velocities, has been given by Hertz, and the result has been verified experimentally<sup>15</sup>.

At any instant during the impact, the quantity  $\alpha$  is the relative displacement of the centers of mass of the two bodies, estimated from their relative positions at the instant when the impact commences, and resolved in the direction of the common normal. The pressure  $P$  between the bodies is the rate of destruction of the momentum of either. We therefore have the equation

$$\frac{d}{dt} \left( m_1 \frac{m_2 \dot{\alpha}}{m_1 + m_2} \right) = -P, \quad (60)$$

where  $\dot{\alpha}$  stands for  $d\alpha/dt$ , and  $m_1, m_2$  are the masses of the bodies<sup>16</sup>. Now  $P$  is a function of  $t$ , so that the principal semi-diameters  $a$  and  $b$  of the compressed area at any instant are also functions of  $t$ , determined in terms of  $P$  by the second and third of equations (54); in fact  $a$  and  $b$  are each of them proportional to  $P^{\frac{1}{3}}$ . Equation (57) shows that  $\alpha$  is proportional to  $P^{\frac{2}{3}}$ , or that  $P$  is proportional to  $\alpha^{\frac{3}{2}}$ ; we write

$$P = k_2 \alpha^{\frac{3}{2}}, \quad (61)$$

where

$$\left( \frac{3}{4} \right)^2 k_2^2 A(\theta_1 + \theta_2)^2 \left[ \int_0^\infty \frac{d\zeta}{\sqrt{(1+\zeta)(1+\zeta-e^2)\zeta}} \right]^3 = \int_0^\infty \frac{d\zeta}{(1+\zeta)\sqrt{(1+\zeta)(1+\zeta-e^2)\zeta}} \quad (62)$$

Equation (60) may now be written

$$\ddot{\alpha} = -k_1 k_2 \alpha^{\frac{3}{2}}, \quad (63)$$

where  $k_1 = (m_1 + m_2)/m_1 m_2$ . This equation may be integrated in the form

$$\frac{1}{2}(\dot{\alpha}^2 - v^2) = -\frac{2}{5} k_1 k_2 \alpha^{\frac{5}{2}}, \quad (64)$$

<sup>13</sup>This picture is not reproduced here.

<sup>14</sup>Hertz, *J. f. Math. (Crelle)*, Bd. 92 (1881), reprinted in Hertz, *Ges. Werke*, Bd. 1, p. 170.

<sup>15</sup>Schneebeli, *Arch. des sci. phys., Geneva*, t. 15 (1885). Investigations of the duration of impact in the case of high velocities were made by Tait, *Edinburgh Roy. Soc. Trans.*, vols. 36, 37 (1890, 1892), reprinted in P.G.Tait, *Scientific Papers*, vol. 2, Cambridge 1900, pp. 222, 249. The theory will be discussed in Chapter XX *infra*.

<sup>16</sup>Assuming that the first body is on the left with respect to the second one, we write the equations of motion of the mass centers in the form

$$m_1 \ddot{x}_1 = -P, \quad m_2 \ddot{x}_2 = P.$$

Denoting by  $d_1$  and  $d_2$  the distances from the centers of mass  $x_1, x_2$  to the contact point, we have from geometrical considerations  $x_2 = x_1 + d_1 + d_2 - \alpha$ . Eliminating from the equations of motion both  $x_1$  and  $x_2$ , we obtain (60).

where  $v$  is the initial value of  $\dot{\alpha}$ , i.e., the velocity of approach of the bodies before impact. The value of  $\alpha$  at the instant of greatest compression<sup>17</sup> is

$$\left(\frac{5}{k_1 k_2}\right)^{\frac{2}{5}} \left(\frac{v}{2}\right)^{\frac{4}{5}}; \quad (65)$$

and, if this quantity is denoted by  $\alpha_1$ , the duration of the impact is<sup>18</sup>

$$2 \int_0^{\alpha_1} \frac{d\alpha}{\left[v^2 - \frac{4}{5}k_1 k_2 \alpha^{\frac{5}{2}}\right]^{\frac{1}{2}}},$$

which is<sup>19</sup>

$$2 \frac{\alpha_1}{v} \int_0^1 \frac{dx}{(1-x^{\frac{5}{2}})^{\frac{1}{2}}} = \frac{4}{5} \sqrt{\pi} \frac{\alpha_1}{v} \frac{\Gamma(\frac{2}{5})}{\Gamma(\frac{9}{10})} = 2.9432 \frac{\alpha_1}{v}.$$

We may express  $\alpha_1$  in terms of the shapes and masses of the bodies and the velocities of propagation of waves of compression in them; let  $V_1$  and  $V_2$  be these velocities<sup>20</sup>,  $\rho_1$  and  $\rho_2$  the densities of the bodies,  $\nu_1$  and  $\nu_2$  the values of Poisson's ratio for the two materials; then

$$\theta_1 = \frac{(1-\nu_1)^2}{\pi V_1^2 \rho_1 (1-2\nu_1)}, \quad \theta_2 = \frac{(1-\nu_2)^2}{\pi V_2^2 \rho_2 (1-2\nu_2)}, \quad (66)$$

so that

$$\alpha_1 = \left[ \frac{5m_1 m_2 v^2}{4(m_1 + m_2)} \frac{3\sqrt{A}}{4\pi} \left\{ \frac{(1-\nu_1)^2}{V_1^2 \rho_1 (1-2\nu_1)} + \frac{(1-\nu_2)^2}{V_2^2 \rho_2 (1-2\nu_2)} \right\} I \right]^{\frac{5}{2}}, \quad (67)$$

where

$$I^2 \int_0^\infty \frac{d\zeta}{(1+\zeta)\sqrt{(1+\zeta)(1+\zeta-e^2)\zeta}} = \left[ \int_0^\infty \frac{d\zeta}{\sqrt{(1+\zeta)(1+\zeta-e^2)\zeta}} \right]^3. \quad (68)$$

It appears that the duration of the impact varies inversely as the fifth root of the relative velocity of approach before impact. The order of magnitude of the gravest period of free vibration that would involve compression is  $1/A_1 V_1$ , and thus the duration of impact bears to this period a ratio of which the order of magnitude is  $(V_1/v)^{\frac{1}{5}}$ .

## 140. Impact of spheres

When the bodies are spheres of radii  $r_1, r_2$ , we have

$$\begin{aligned} A = B &= \frac{1}{2} \left( \frac{1}{r_1} + \frac{1}{r_2} \right), \quad e = 0, \quad a = b, \\ a^3 &= \frac{3\pi}{4} \frac{r_1 r_2}{r_1 + r_2} (\theta_1 + \theta_2) P, \\ \alpha &= \frac{3\pi}{4a} (\theta_1 + \theta_2) P; \end{aligned} \quad (69)$$

from which we find

$$\begin{aligned} k_2 &= \frac{4}{3\pi} \left( \frac{r_1 r_2}{r_1 + r_2} \right)^{\frac{1}{2}} \frac{1}{\theta_1 + \theta_2}, \quad a = \left( \frac{\alpha r_1 r_2}{r_1 + r_2} \right)^{\frac{1}{2}}, \\ \alpha_1 &= \left[ \frac{15\pi v^2 (\theta_1 + \theta_2) m_1 m_2}{16(m_1 + m_2)} \right]^{\frac{2}{5}} \left( \frac{r_1 + r_2}{r_1 r_2} \right)^{\frac{1}{5}}. \end{aligned} \quad (70)$$

Hence the duration of the impact and the radius of the (circular) compressed area are determined.

<sup>17</sup>When  $\dot{\alpha} = 0$ .

<sup>18</sup>Obtained after resolving (64) with respect to  $dt$  and integration.

<sup>19</sup>After the substitution of variable  $\alpha = \alpha_1 x$  and in view of (65).

<sup>20</sup> $V_1^2$  is  $(\lambda_1 + 2\mu_1)/\rho_1$  and  $V_2^2$  is  $(\lambda_2 + 2\mu_2)/\rho_2$ . See Chapter XIII *infra*.

In the particular case of equal spheres of the same material the duration of the impact is

$$(2.9432\dots) \left\{ \frac{25\pi^2}{8} \frac{(1-\nu)^4}{(1-2\nu)^2} \right\}^{\frac{1}{5}} \frac{r}{v^{\frac{1}{5}} V^{\frac{4}{5}}}, \quad (71)$$

where  $r$  is the radius of either sphere,  $\nu$  is the Poisson's ratio of the material, and  $V$  is the velocity of propagation of waves of compression. The radii of the circular patches that come into contact are each equal to

$$r \left( \frac{v}{V} \right)^{\frac{2}{5}} \left[ \frac{5\pi^2}{16} \frac{(1-\nu)^2}{1-2\nu} \right]^{\frac{1}{5}}. \quad (72)$$

These results have been verified experimentally<sup>21</sup>.

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<sup>21</sup>Schneebeli, *Rep. d. Phys.*, Bd. 22 (1886), and Hamburger, *Tageblatt d. Nat. Vers. in Wiesbaden*, 1887.