1. (a) Use the definition of the derivative (that is, use the limit definition that we gave in lecture 21, and do not use the product/chain/quotient rules from lecture 22) to calculate the derivative $f'(x)$ of $f(x) := x + c$, where c is an arbitrary constant.
(b) Use the derivative you calculated in part (a), along with the product rule, to differentiate the function g given by $g(x) := (x + 1)(x + 2)$
(c) Finally, check that your answer is correct by expanding the formula for $g(x)$ and then differentiating. (You may use any differentiation techniques you want for this part.)

Solution:

(a) The definition of the derivative is:

$$f'(x) = \lim_{\Delta x \to \infty} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Find the derivative using this formula:

$$f'(x) = \lim_{\Delta x \to \infty} \frac{(x + \Delta x + c) - (x + c)}{\Delta x} = \lim_{\Delta x \to \infty} \frac{x + \Delta x + c - x - c}{\Delta x} = \lim_{\Delta x \to \infty} \frac{\Delta x}{\Delta x} = 1$$

(b) $g'(x) := (x + 1)'(x + 2) + (x + 1)(x + 2)' = 1 \cdot (x + 2) + (x + 1) \cdot 1 =$

$$= x + 2 + x + 1 = 2x + 3$$

(c) $(x + 1)(x + 2) = x^2 + 2x + x + 2 = x^2 + 3x + 2$

Then $g'(x) = (x^2 + 3x + 2)' = (x^2)' + 3(x)' + (2)' = 2x + 3 \cdot 1 = 2x + 3$

The answers in (b) and (c) is equal and correct.

Answer: $f'(x) = 1, g'(x) = 2x + 3$

2. Let f and g be differentiable functions with $g(x) \neq 0$ for all x. Use the product and chain rules to differentiate the function h given by $h(x) := \frac{f(x)}{g(x)}$ (Do not use the quotient rule to solve this problem. As a hint, note that $\frac{f(x)}{g(x)} = f(x) \cdot \left(g(x)\right)^{-1}$)

Solution:

$$h(x) = f(x) \cdot \left(g(x)\right)^{-1}$$

Then find the derivative using the product rule:

$$h'(x) = f'(x) \cdot \left(g(x)\right)^{-1} + f(x) \cdot \left(\left(g(x)\right)^{-1}\right)' = \frac{f'(x)}{g(x)} + f(x) \cdot \left(-1 \cdot \left(g(x)\right)^{-2} \cdot g'(x)\right) =$$

$$= \frac{f'(x)}{g(x)} - f(x) \cdot \left(g(x)\right)^{-2} \cdot g'(x) = \frac{f'(x)}{g(x)} - \frac{f(x) \cdot g'(x)}{\left(g(x)\right)^2} = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{\left(g(x)\right)^2}$$

Answer: $h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{\left(g(x)\right)^2}$

3. The lemniscate of Gerono (or Huygens) is the curve given by

$$x^4 - x^2 + y^2 = 0$$

(a) Show that if a point (a, b) is on this curve, then so are $(-a, b), (a, -b)$ and $(-a, -b)$.

(b) Sketch this curve in the x–y plane.

(c) Use implicit differentiation to find the slope of this curve (when it exists).

(d) Show that the point $c = \left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ lies on this curve and calculate the tangent line at this point.

Solution:

(a) If a point (a, b) is on this curve then:

$$a^4 - a^2 + b^2 = 0$$

Put the coordinates $(-a, b)$ into the expression of the left side of equation of the lemniscate:

$$(-a)^4 - (-a)^2 + b^2$$

Since $(-a)^{2n} = a^{2n}$, then rewrite the expression:

$$(-a)^4 - (-a)^2 + b^2 = a^4 - a^2 + b^2 = 0.$$

The point $(-a, b)$ is on the curve.

Put the coordinates $(a, -b)$ into the expression of the left side of equation of the lemniscate:

$$a^4 - a^2 + (-b)^2 = a^4 - a^2 + b^2 = 0$$

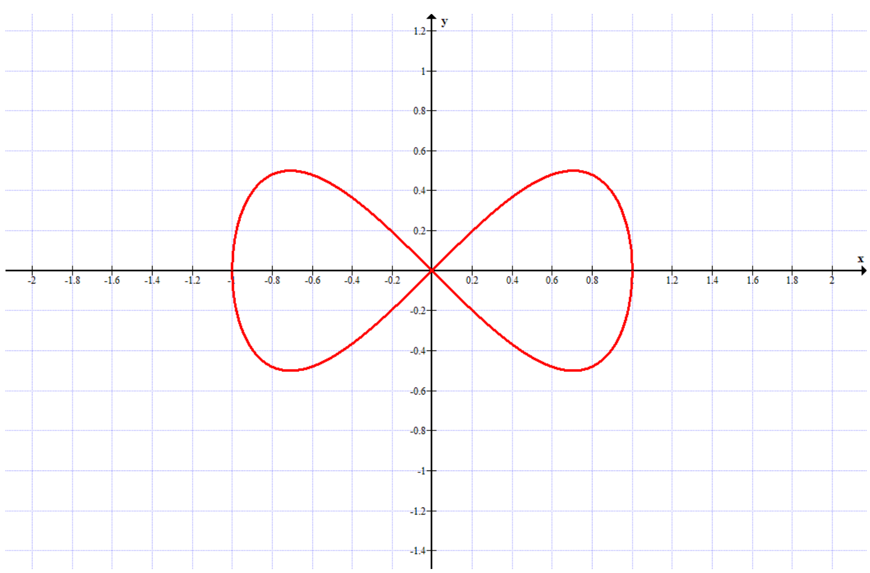
The point $(a-, b)$ lies on the curve.

Put the coordinates $(-a, -b)$ into the expression of the left side of equation of the lemniscate:

$$(-a)^4 - (-a)^2 + (-b)^2 = a^4 - a^2 + b^2 = 0$$

The point $(-a, -b)$ lies on the curve.

(b)



(c) Take d/dx of both sides of the equation:

$$\frac{d}{dx}(x^4 - x^2 + y^2) = \frac{d}{dx}(0)$$
$$4x^3 - 2x + 2yy' = 0$$

Solve for y':

$$2yy' = 2x - 4x^3$$
$$y' = \frac{2x - 4x^3}{2y}$$
$$y' = \frac{x - 2x^3}{y}$$

The slope of the curve is equal to the derivative of the function: $m = \frac{x-2x^3}{y}$

(d) Put the coordinates of the point into the equation:

$$\left(\frac{1}{\sqrt{2}}\right)^4 - \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{4} - \frac{1}{2} + \frac{1}{4} = 0$$

The point $c = \left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ lies on this curve.

Calculate the tangent line at this point. First find the slope:

$$m = \frac{\frac{1}{\sqrt{2}} - 2 \cdot \left(\frac{1}{\sqrt{2}}\right)^3}{\frac{1}{2}} = \frac{\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{8}}}{\frac{1}{2}} = \frac{\frac{1}{\sqrt{2}} - \frac{2}{2\sqrt{2}}}{\frac{1}{2}} = \frac{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}}{\frac{1}{2}} = 0$$

Then find the equation of the line using point slope form:

$$y - y_1 = m(x - x_1)$$
$$y - \frac{1}{2} = 0\left(x - \frac{1}{\sqrt{2}}\right)$$
$$y - \frac{1}{2} = 0$$
$$y = \frac{1}{2}$$

The tangent line at a point $c = \left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ is $y = \frac{1}{2}$

Answer: the slope is $m = \frac{x-2x^3}{y}$, the tangent line at a point $c = \left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ is $y = \frac{1}{2}$

4. Define a function f on $\mathbb{R}$ by $f(x) := x^3 e^x$

(a) Sketch the graphs of $y = x^3$, $y = e^x$ and $y = f(x)$. You may assume that $\lim_{x \to -\infty} f(x) = 0$ when drawing this graph.

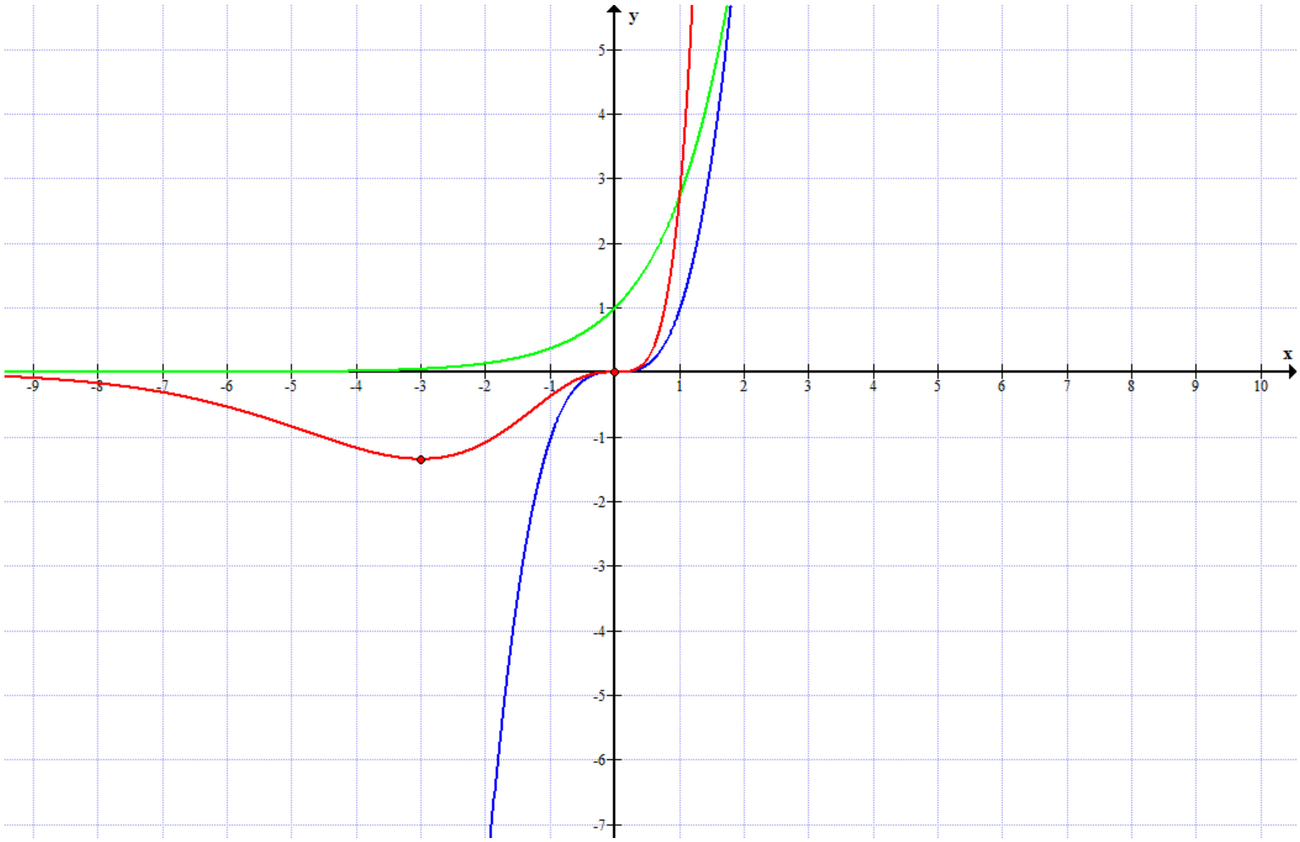
(b) Are there any critical points of f which are not stationary points?

(c) Find the critical points of f, and determine their nature from your sketch.

(d) Find the global maxima and minima of f.

Solution:

(a) The picture shows the curve $y = x^3$ graph in blue, the curve $y = e^x$ graph in green and the curve $y = x^3 e^x$ graph in red.



(b) There are no critical points of f which are not stationary points, because the function is differentiable and there are no points where the derivative is not defined

(c) Find the derivative of the function using the product rule:

$$f'(x) = (x^3 e^x)' = (x^3)' e^x + x^3 (e^x)' = 3x^2 e^x + x^3 e^x$$

Set the $f'(x) = 0$. Then

$$3x^2 e^x + x^3 e^x = 0$$
$$x^2 e^x (3 + x) = 0$$

Since $e^x \neq 0$, then

$$x^2 = 0 \text{ or } 3 + x = 0$$
$$x = 0, x = -3$$

$x = -3$ is the minimum point, $x = 0$ is the point of horizontal inflection

(d) The global minima is $f(-3) = (-3)^3 e^{-3} = -\frac{27}{e^3} \approx -1{,}34$. The global maxima is not exist.

Answer: $x = -3$ is the minimum point, $x = 0$ is the point of horizontal inflection, the global minima is $-1{,}34$. The global maxima is not exist.

5. (a) Use the first and second derivative tests to find and classify the critical points of the function f given by $f(x) := e^{x^2}$

(b) Find and classify the critical points of the function g given by $g(x) := x^4$

Solution:

(a) Find the first derivative of the function f:

$$f'(x) = \left(e^{x^2}\right)' = e^{x^2} \cdot (x^2)' = e^{x^2} \cdot 2x = 2xe^{x^2}$$

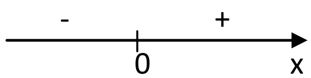
Find the critical points ($f'(x) = 0$):

$$2xe^{x^2} = 0$$

Since $e^{x^2} \neq 0$, then

$$x = 0 \text{ - the critical point}$$

Use the first derivative test:



The f has a local minimum at $x = 0$.

Use the second derivative test. Find the second derivative of the function f:

$$f''(x) = \left(2xe^{x^2}\right)' = (2x)'e^{x^2} + 2x\left(e^{x^2}\right)' = 2e^{x^2} + 2xe^{x^2} \cdot (x^2)' = 2e^{x^2} + 2xe^{x^2} \cdot 2x =$$
$$= 2e^{x^2} + 4x^2e^{x^2}$$

Then find the $f''(0)$:

$$f''(0) = 2e^{0^2} + 4 \cdot 0^2 e^{0^2} = 2 \cdot 1 + 0 = 2 > 0$$

Thus f has a local minimum at $x = 0$

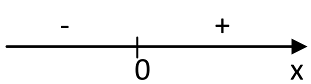
(b) Find the first derivative of the function g:

$$g'(x) = (x^4)' = 4x^3$$

Find the critical points ($g'(x) = 0$):

$$4x^3 = 0$$
$$x = 0 \text{ - the critical point}$$

Use the first derivative test:



The g has a local minimum at $x = 0$.

Use the second derivative test. Find the second derivative of the function g:

$$f''(x) = (4x^3)' = 4 \cdot 3x^2 = 12x^2$$

Then find the $f''(0)$:

$$f''(0) = 12 \cdot 0^2 = 0$$

The second derivative test is not informative.

Answer: f has a local minimum at $x = 0$, g has a local minimum at $x = 0$.

6. (a) Sketch the graphs of the functions f and g given by

$$f(x) := \sqrt{x}, \qquad g(x) := x^2$$

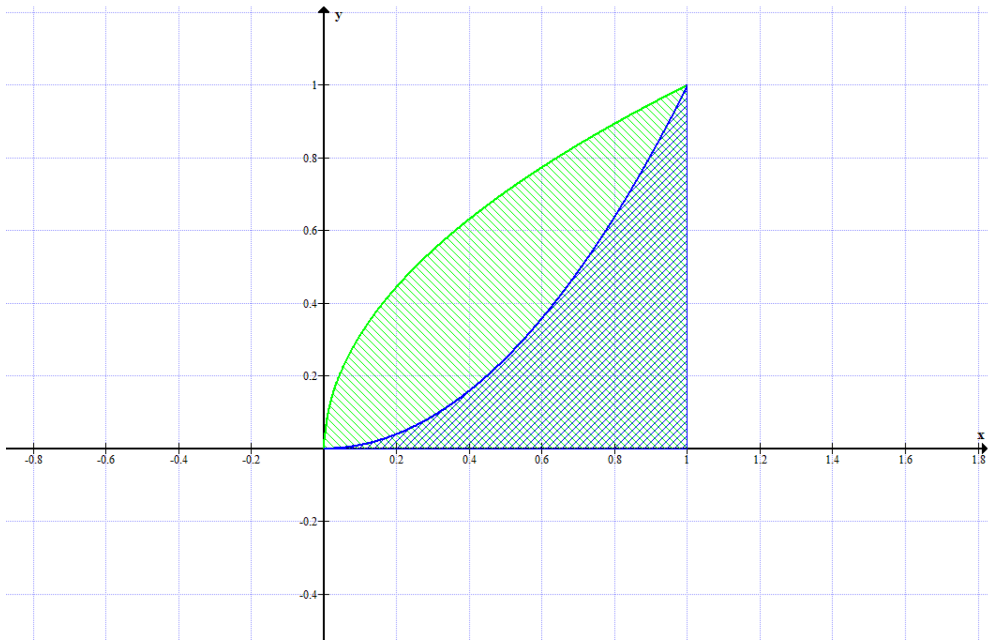
over the interval $0 \le x \le 1$, and calculate the areas under these graphs.

(b) Calculate the (definite) integral

$$\int_{-42}^{42} \frac{xe^{x^2}}{x^2+1}\,dx$$

Solution:

(a) The picture shows the curve $y = x^2$ graph in blue, the curve $y = \sqrt{x}$ graph in green



Find the area under $f(x)$:

$$S = \int_0^1 f(x)dx = \int_0^1 \sqrt{x}\,dx = \frac{2}{3}\sqrt{x^3}\Big|_0^1 = \frac{2}{3}\sqrt{1^3} - \frac{2}{3}\sqrt{0^2} = \frac{2}{3}$$

Find the area under $g(x)$:

$$S = \int_0^1 g(x)dx = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

(b) Show that the function $f(x) = \frac{xe^{x^2}}{x^2+1}$ is odd:

$$f(-x) = \frac{-xe^{(-x)^2}}{(-x)^2+1} = \frac{-xe^{x^2}}{x^2+1} = -\frac{xe^{x^2}}{x^2+1} = -f(x)$$

This means that *f*(*x*) is odd.

Since $\int_{-a}^{a} f(x)\,dx = 0$ for odd function, then

$$\int_{-42}^{42} \frac{xe^{x^2}}{x^2+1}dx = 0$$

Answer:

(a) $\frac{2}{3}, \frac{1}{3}$

(b) 0

7. Find the following (indefinite) integrals

(a) $\int (2x^2+1)e^x dx$

(b) $\int x^3\sqrt{x^4+1}dx$

Solution:

(a) Use the integration by parts:

$$u = 2x^2+1, \qquad du = 4xdx$$
$$dv = e^x dx, \qquad v = e^x$$

Then $\int udv = uv - \int vdu$:

$$\int (2x^2+1)e^x dx = (2x^2+1)e^x - \int e^x \cdot 4x\,dx = (2x^2+1)e^x - 4\int xe^x dx$$

Use the the integration by parts:

$$u = x, \qquad du = dx$$
$$dv = e^x dx, \qquad v = e^x$$

Then

$$\int (2x^2+1)e^x dx = (2x^2+1)e^x - 4\left(xe^x - \int e^x dx\right) = (2x^2+1)e^x - 4(xe^x - e^x) + C =$$
$$2x^2e^x + e^x - 4xe^x + 4e^x + C = 2x^2e^x - 4xe^x + 5e^x + C = (2x^2-4x+5)e^x + C$$

(b) Use the substitution method:

$$x^4+1 = t$$
$$dt = 4x^3dx$$

$$\int x^3\sqrt{x^4+1}dx = \frac{1}{4}\int 4x^3\sqrt{x^4+1}dx = \frac{1}{4}\int \sqrt{t}\,dt = \frac{1}{4}\cdot\frac{2}{3}\sqrt{t^3} + C = \frac{\sqrt{t^3}}{6} + C =$$
$$= \frac{\sqrt{(x^4+1)^3}}{6} + C$$

Answer:

(a) $\int (2x^2+1)e^x dx = (2x^2-4x+5)e^x + C$

(b) $\int x^3\sqrt{x^4+1}dx = \frac{\sqrt{(x^4+1)^3}}{6} + C$